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A Multilevel Variational Method for $A\mathbf{u} = \lambda B\mathbf{u}$ on Composite Grids*

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There are two existing general approaches to the numerical solution of differential eigenproblems by multigrid, one which uses a linear multigrid solver for the inner loop of a linearization technique (e.g., inverse iteration) and the other which integrates multigrid directly into the problem solution (e.g., FAS). The present paper developes a third multigridlike approach, RQMG, that appeals directly to the variational problem of minimizing the Rayleigh quotient, applying easily to both global and local grid implementations. Numerical results are presented for a simplified single group diffusion problem.

1. INTRODUCTION

Suppose A and B are real symmetric matrices on a finite-dimensional Euclidean space H with innerproduct $\langle \cdot, \cdot \rangle$. We have in mind that A and B result from discretization of an elliptic differential eigenproblem. Consider the problem of finding the smallest eigenvalue and a unit eigenvector for the problem

$$A\mathbf{u} = \lambda B\mathbf{u}, \qquad \langle \mathbf{u}, B\mathbf{u} \rangle = 1, \qquad \mathbf{u} \in H, \lambda \text{ real.} \tag{1.1}$$

(Solutions of equations will be denoted by bold letters.) Then there are two general multigrid approaches that have been studied for numerical solution of (1.1), namely:

Multigrid as a linear solver. The discrete equations in (1.1) can be linearized by methods such as inverse iteration with linear multigrid solvers used for the inner loop. With care, multigrid can work very effectively here even though the linearized equations are indefinite and illconditioned. Many authors have studied this approach (cf., the early work in [1, 7, 13]).

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Multigrid applied directly to the equations. Schemes such as those described in [2, 4] more fully integrate multigrid principles into the problem solution by applying multigrid directly to the nonlinear system (1.1). For general nonlinear problems, such methods tend to be somewhat more effective than the linear solver approach. However, when carefully implemented, there is really very little difference in the performance of these two approaches as they apply to eigenproblems because such nonlinearity is "mild."

The method introduced in [5] can be thought of as a combination of the above two approaches because it applies multigrid principles directly to (1.1) using a linearization-like technique. See Section 5 for further details.

The method treated in this paper takes the different approach of applying multigrid principles directly to the variational problem

$$RQ(\mathbf{u}) = \min_{Bu \neq 0} RQ(u), \tag{1.2}$$

where $RQ(u) = \langle Au, u\rangle / \langle Bu, u\rangle$ is the Rayleigh quotient. This is done by way of transferring the fine grid functional **exactly** to the coarser levels and using some relaxation scheme like coordinate relaxation [3] that applies to such functionals. First described in [9], we refer here to this technique as Rayleigh quotient minimization multigrid, or RQMG. As we shall see, RQMG needs very little care concerning such mechanisms as shifts and normalization conditions, and it has several advantages over the orther approaches. We will also see that RQMG extends easily to the case where local grids are used; that is, it provides a natural foundation for the fast adaptive composite grid method (FAC) [6, 11].

This work was inspired by the efforts at Los Alamos National Laboratory to develop a multigrid-type scheme for solving the eigenproblem that arises in the single group neutron diffusion problem. The model problem here is the 2D elliptic equation

$$\begin{aligned} -\Delta \mathbf{u} &= \lambda f \mathbf{u} \\ \int_{\Omega} \mathbf{u}^2 \, d\Omega &= 1, \end{aligned} \tag{1.3}$$

where $\Omega = [0, 1]^2$ is the unit square and f is a function with support $\Omega_1 = [\alpha, \beta]^2$, $0 \leqslant \alpha \leqslant \beta \leqslant 1$. For example,

$$f(x, y) = \begin{cases} 1, & \alpha \leqslant x, \quad y \leqslant \beta \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

(A model of more practical interest would, in place of Δ in (1.3), have a diffusion operator with discontinuous coefficients. However, the model as it is has the essential features of interest to us here and was in fact the cause of some difficulties with the other two approaches.) This model is of particular importance because of the

need to use local grids over Ω_1. We therefore include a numerical study of RQMG applied to (1.3) using both global and local grids.

In this paper we keep discussion, notation, and development of RQMG to a minimum. In fact, after introducing the notation (Section 2), we present RQMG in its unigrid [10] form (Section 3) and then briefly describe how it is implemented in a practical multigrid-like framwork (Section 4). We then discuss a linearized method (Section 5) and finish with a report on numerical experiments (Section 6). A theoretical analysis of the RQMG algorithm will be presented elsewhere.

Unigrid is a very simple way to describe, implement, and test variationally formulated multigrid-like algorithms. In fact, the resulting algorithm performance is **exactly** what would be achieved by a more involved practical implementation except that the computational cost is much greater. The point here is that we can quickly implement and exactly predict the performance of various multigrid algorithms without the attendant programming troubles. This is best illustrated by noting that developing the code for the global grid solution of (1.3) together with collecting the results took a total of just a few man hours of effort. In fact, modification of the code to accommodate local grids took just a few additional minutes.

2. Notation and Assumptions

Since (1.1) is to be thought of as a discretization of some differential eigenproblem, and because multigrid methods require several coarser versions of such problems, we assume that (1.1) actually represents a family of discrete eigenproblems parameterized by h, the "mesh size." We thus write

$$A^h\mathbf{u} = \lambda^h B^h\mathbf{u}^h \qquad \mathbf{u}^h \in H^h,\ \lambda^h \text{ real}$$
$$\langle \mathbf{u}^h, B^h\mathbf{u}^h \rangle = 1, \tag{2.1}$$

where H^h is the discrete Euclidean space of grid functions defined on grid Ω^h equipped with the innerproduct $\langle \cdot, \cdot \rangle$, A^h, B^h: $H^h \to H^h$ are symmetric linear operators and B^h is positive semi-definite, and $\mathbf{u}^h$ denotes **an** exact eigenvector associated with **the** smallest eigenvalue λ^h. Approximations will be denoted by u^h and λ^h, respectively. With a sequence of m parameters $0 < h_m < h_{m-1} < \cdots < h_1$, we are given a sequence of increasingly finer global grids $\Omega^1, \Omega^2, \ldots, \Omega^m$, and their vector spaces $H^1, H^2, \ldots, H^m$. (k will replace h_k where appropriate. For the finest level Ω^m, we will frequently drop h_m and m altogether. This should not cause confusion with continuous function space quantities which are in script notation.) Thus, the variational formulation of (2.1) for $h = h_m$ is

$$RQ(\mathbf{u}) = \min_{u \neq 0} \frac{\langle Au, u \rangle}{\langle Bu, u \rangle}. \tag{2.2}$$

The intergrid transfers are denoted by $I_k^j : H^k \to H^j$, $1 \leqslant k, j \leqslant m$. We assume

they are linear and full rank and refer to the case $j>k$ as prolongation and $j<k$ as restriction. We assume $I_k^k = I$, the identity operator on H^k, and that $I_k^i = I_j^i I_k^j$ for j between i and k.

On each level k we let $\{d_i^k : 1 \leqslant i \leqslant n^k\}$ denote the canoncial basis for H^k (i.e., d_i^k is the ith column of the identity matrix for H^k). Let the set S be the collection of all basis vectors mapped to H^m, that is, $S = \{I_k^m d_i^k : 1 \leqslant i \leqslant n^k,\ 1 \leqslant k \leqslant m\}$.

To avoid iteration subscripts, we use u^k as **dynamic** approximations and denote iterations by the replacement expression $u^k \leftarrow v^k$. This means that u^k is replaced by v^k which may itself be an expression involving (the old) u^k. Unless otherwise subscripted, λ^h will denote the smallest eigenvalue of (2.1). We refer to any nonzero vector $\mathbf{u}^h$ that satisfies $A^h\mathbf{u}^h = \lambda^h B^h\mathbf{u}^h$ for some λ^h as an eigenvector of $(A^h; B^h)$. We let $\rho(C)$ denote the spectral radius of a symmetric matrix C and $\text{cond}(C) = \rho(C)\,\rho(C^{-1})$.

3. Unigrid Formulation

In this section we develop RQMG in its unigrid form because it greatly simplifies both the explanation and the theory. If the practical definition of RQMG is preferred, it may be best to skip to Section 4.

It is first important to understand unigrid as a method for solving a symmetric, positive definite linear equation $C\mathbf{v} = f$. Briefly (see [10] for more detail), the idea is to solve this system by minimizing the quadratic energy functional $F(v) = \langle Cv, v\rangle - 2\langle v, f\rangle$. Thus, one "fine-to-coarse" cycle of unigrid with some initial approximation v begins by subtracting from v the multiple of the coordinate function d_i^m that minimizes F along this direction. Called coordinate relaxation [3], this process proceeds in turn as i varies in some ordering over $\{1, 2, ..., n^m\}$. This is then followed by the same process but with d_i^m replaced by $I_{m-1}^m d_i^{m-1}$, then again with $I_{m-1}^m d_i^{m-1}$ replaced by $I_{m-2}^m d_i^{m-2}$, and so on to include $I_1^m d_i^1$. In other words, one unigrid cycle proceeds by selecting d in turn from S and making the replacement $v \leftarrow v - s^* d$, where s^* minimizes $F(v - sd)$ over s.

To see why unigrid is equivalent to conventional multigrid, note that coordinate relaxation on the energy functional is equivalent to Gauss–Seidel on the linear system and that the steps involving d_i^k are equivalent to Gauss–Seidel steps on the coarse grid equations $I_m^k C I_k^m \mathbf{v}^k = I_m^k (f - Cv)$ with unknown $\mathbf{v}^k$, followed by the correction $v \leftarrow v + I_k^m v^k$.

The extension of unigrid to the eigenproblem (1.2) is now direct. It is simply a matter of recognizing that the minimizer, s^*, of $RQ(u - sd)$ can be computed exactly for any d as a particular real zero of a quadratic polynomial in s. (See Section 4.) Thus, as in the linear case, one cycle of the unigrid form of RQMG for solving (2.2) proceeds by selecting $d = I_k^m d_i^k$ in turn in some ordering and performing the step

$$\begin{aligned} RQ(u - s^* d) &= \min_s RQ(u - sd) \\ u &\leftarrow u - s^* d. \end{aligned} \tag{3.1}$$

That is, s^* is chosen to minimize $RQ(u - sd)$ and u is then replaced by $u - sd$. For example, to simulate a lexicographic $V(0, 1)$-cycle [12], we use the index ordering suggested by the Fortran loops:

```
      D0    10    K = 0, M
      D0    10    I = 1, N1(K)
      D0    10    J = 1, N2(K)
10    D(I, J) = FN(I, J, K)
```

Actually, for the numerical results reported in Section 5, we used a red–black ordering within each level.

Local refinements are implemented simply by adding a new level of finer grid basis functions with support in the refinement region. More precisely, suppose level k is to be restricted to a particular subregion. Then the level k steps with $d = I_k^m d_i^k$ in (3.1) are simply restricted to those i that correspond to grid points of the given subregion.

To anticipate how this formulation can be made practical, note that for fixed $k < m$, as d sweeps over $I_k^m d_i^k$, (3.1) can be interpreted as a simple coordinate relaxation method for solving the level k variational problem given by

$$RQ(u - I_k^m \mathbf{v}^k) = \min_{v^k} RQ(u - I_k^m \mathbf{v}^k). \tag{3.2}$$

It is now easy to see that such a sweep involves level k quantities only.

4. Practical RQMG

The unigrid scheme just described is computationally very expensive. This is primarily because all relaxation steps are performed on level Ω^m. However, these computations can in fact be done **exactly** on their respective grids just as multigrid is done for the linear case. To see this, we first need to define the coarse grid operators A^k and B^k that are induced by the variational formulation. They are specified by the **Galerkin condition**

$$\begin{aligned} A^k &= I_{k+1}^k A^{k+1} I_k^{k+1}, \\ B^k &= I_{k+1}^k B^{k+1} I_k^{k+1}, \qquad 0 \leqslant k \leqslant m-1. \end{aligned} \tag{4.1}$$

The grid transfers are assumed to satisfy the **energy condition**

$$I_{k+1}^k = \xi^k I_k^{k+1^T}, \tag{4.2}$$

where ξ^k is some positive scale factor. For simplicity and without loss of generality, we henceforth assume $\xi^k = 1$.

The following describes one $V(1, 1)$-cycle of a practical multigrid algorithm that is in fact theoretically equivalent to a corresponding unigrid scheme as described in the previous section. We first write for $u \in H^m$

$$RQ(u - sI_k^m d_i^k) = \frac{\langle u, A^m u\rangle - 2s\langle I_m^k A^m u, d_i^k\rangle + s^2 A_{ii}^k}{\langle u, B^m u\rangle - 2s\langle I_m^k B^m u, d_i^k\rangle + s^2 B_{ii}^k},$$

where

$$A_{ii}^k = \langle d_i^k, A^k d_i^k\rangle,$$

and

$$B_{ii}^k = \langle d_i^k, B^k d_i^k\rangle.$$

Keeping track of the quantities $a = \langle u, A^m u\rangle$, $b = \langle u, B^m u\rangle$, $I_m^k A^m u$, and $I_m^k B^m u$ dynamically yields the following algorithm. Note that the coarse grid u^k, $1 \leqslant k \leqslant m-1$, approximate corrections to u^m and not $\mathbf{u}^k$, the coarse grid eigenvectors.

Step 1. Perform one sweep of coordinate relaxation using u^m.

Step 2. Set $q^m \leftarrow 0$, $r^m \leftarrow 0$, $a \leftarrow \langle A^m u^m, u^m\rangle$, and $b \leftarrow \langle B^m u^m, u^m\rangle$.

Step 3. For $k = m-1$ to 1 step -1, do the following: $q^k \leftarrow I_{k+1}^k (q^{k+1} + A^{k+1} u^{k+1})$; $r^k \leftarrow I_{k+1}^k (r^{k+1} + B^{k+1} u^{k+1})$; $u^k \leftarrow 0$; and for $i = 1$ to n^k do the following relaxation scheme (which includes updating a and b);

Compute the minimizer, s^*, that satisfies

$$RQ^k(u^k - s^* d_i^k) = \min_s RQ^k(u^k - s d_i^k), \tag{4.3}$$

where

$$RQ^k(u^k - s d_i^k) = \frac{a - 2s(q_i^k + \alpha_i^k) + s^2 A_{ii}^k}{b - 2s(r_i^k + \beta_i^k) + s^2 B_{ii}^k}, \tag{4.4}$$

$$q_i^k = \langle q^k, d_i^k\rangle, \qquad r_i^k = \langle r^k, d_i^k\rangle,$$

$$A_{ii}^k = \langle A^k d_i^k, d_i^k\rangle, \qquad B_{ii}^k = \langle B^k d_i^k, d_i^k\rangle,$$

$$\alpha_i^k = \langle A^k u^k, d_i^k\rangle, \quad \text{and} \quad \beta_i^k = \langle B^k u^k, d_i^k\rangle;$$

$u^k \leftarrow u^k - s^* d_i^k$;

$a \leftarrow a - 2s^*(q_i^k + \alpha_i^k) + s^{*2} A_{ii}^k$;

and

$b \leftarrow b - 2s^*(r_i^k + \beta_i^k) + s^{*2} B_{ii}^k$.

Step 4. For $k = 2$ to $m-1$, do the following: $u^k \leftarrow u^k + I_{k-1}^k u^{k-1}$; $u^{k-1} \leftarrow 0$; and for $i = 1$ to n^k perform the same relaxation on u_i^k as in Step 3 (updating a and b accordingly).

Step 5. $u^m \leftarrow u^m + I_{m-1}^m u^{m-1}$.

Step 6. Perform one sweep of coordinate relaxation using u^m.

Note that a and b are global quantities, passed on between levels.

The key to seeing the equivalence here is to note that $RQ^k(u^k) = RQ(u)$, where

$u = u^m + \sum_{j=1}^{m-1} I_j^m u^j$ is the fine grid approximation that would result from immediate correction by each coarse grid computation. Note that all computations involving d_i^k are with level k quantities so that the total cost for $k < m$ is small compared to that for $k = m$, just as it is for linear multigrid methods.

Computation of s^* in (4.5) is quite simple because the critical points for the quotient of two quadratic polynomials in s are just the roots of another quadratic polynomial. In fact, a little analysis of (4.4) shows that if

$$RQ(u) = \frac{a}{b} < \frac{A_{ii}^k}{B_{ii}^k} = RQ(I_k^m d_i^k)$$

(which is always true in practice), and if we let $x = B_{ii}^k(q_i^k + \alpha_i^k) - A_{ii}^k(r_i^k + \beta_i^k)$, $y = bA_{ii}^k - aB_{ii}^k$, and $z = a(r_i^k + \beta_i^k) - b(q_i^k + \alpha^k)$, then $s^* = -2z/(y + \sqrt{y^2 - 4 \times z})$.

An operation count shows that a straightforward implementation of one coarse grid relaxation sweep requires one square root, $M_+ + N_+ + 12$ adds/subtracts, and $M_\times + N_\times + 16$ multiples/divides per grid point, where $M_+ (M_\times)$ is the number of adds/subtracts (multiplies/divides) per grid point used to form $A^k u^k$ and similarly with $N_+ (N_\times)$ for $B^k u^k$. This can be reduced somewhat to $M_+ + N_+ + 12$ adds/subtracts and $M_\times + N_\times + 10$ multiplies/divides by prescaling A^k and B^k and further yet (for simple stencils) by freezing a and b in various ways (e.g., during red–black half sweeps, over one or more full sweeps, or over a full cycle). A linearized method, which does not require square roots, is described in the next section. However, the method as it stands may represent a small added cost over these other alternatives and it gives optimal results in some sense. In fact, the cost of RQMG beyond matrix multiplies is small and independent of the stencil, it is more robust, and it generally has much better convergence rates. Thus, for moderate to large stencils such as those that arise in 3D finite element discretizations, RQMG may actually be much more efficient than other methods. It also has the advantages of: not requiring any artificial normalization or special eigenvalue transfer conditions; availability of a variational measure (i.e., the Rayleigh quotient) to optimize parameters and step sizes, to provide a sense of optimality and a sound basis for coarsening, and to measure performance; and robustness under certain potentially adverse conditions including boundary or coefficient singularities, local grids, reduced coarse grid approximation orders, loss of regularity in general, and bad correlation between fine and coarse grid eigenvalues.

The focus here is on minimizing the Rayleigh quotient. However, as with linear operators whose variational problem is energy functional minimization, on each level there is a corresponding equation that is actually being solved. That is, in light of (3.2), the relaxation sweeps on level k can be considered as iterative methods for finding a solution $(\mathbf{v}^k, \boldsymbol{\mu}^k)$ of the following nonlinear problem with minimal $\boldsymbol{\mu}^k$:

$$A^k \mathbf{v}^k - \boldsymbol{\mu}^k B^k \mathbf{v}^k = I_m^k A^m u - \boldsymbol{\mu}^k I_m^k B^m u, \qquad \boldsymbol{\mu}^k = RQ(u - I_k^m \mathbf{v}^k). \tag{4.5}$$

Note that $\mathbf{v}^k$ is unique whenever $\boldsymbol{\mu}^k$ is not an eigenvalue of $(A^k; B^k)$. The value of

$\mathbf{u}^k$ is always unique because it is the minimum of $RQ(u - I_k^m v^k)$ over v^k. For $k = m$, however, (4.5) is equivalent to (2.1) with $h = h^k$ but without the normalization condition. We are not really interested in any particular value of the norm of $\mathbf{u}^m$, so we leave this free.

5. Linearization

The purpose of this section is to present a linearized version of the method both as a scheme for practical use, and as a theoretical tool to provide further insight.

Finding the minimizer s^* of $RQ(u - sd)$ is equivalent to solving the linear equation

$$\langle d,\ (A - \mu B)(u - s^* d) \rangle = 0 \tag{5.1}$$

with

$$\mu = \min_s RQ(u - sd).$$

In later stages of the algorithm, this number μ tends to be very close to the first eigenvalue of (A, B). We can therefore expect a similar asymptotic convergence rate when μ in (5.1) is replaced by a fixed value, updated via $\mu = RQ(u)$ before each multigrid cycle. Then every minimization step (3.1) with an approximate value of s^* found from (5.1) turns out to be just a Gauss–Seidel step on the appropriate level for the homogeneous system $Au - \mu Bu = 0$.

We thus get the following simplified iterative algorithm:

Step 1. $\mu \leftarrow RQ(u)$.
Step 2. Perform one cycle of the *linear* multigrid method on the system $Au - \mu Bu = 0$.

A word of caution is in order. We want only to *iterate* on the system $Au - \mu Bu = 0$, not to solve it. Therefore, on the coarsest level, one cannot use a direct solver or too many iterations with a fixed μ. If a better convergence factor on the coarsest level is needed, then one should make sure that the component in the direction of $\mathbf{u}^1$, the principal eigenvector on level 1, is not resolved. In our computational experiments we performed two sweeps on level 1.

Hackbusch's method [5] can be now interpreted in our framework as the above linearized method with projection steps

$$u^k \leftarrow u^k - \frac{\tilde{u}^k < u^k, \tilde{u}^k \rangle}{\langle u^k, \tilde{u}^k \rangle},$$

where $\tilde{u}^k$ is an approximation to $\mathbf{u}^k$, the eigenvector at level k, added at certain stages of the linear multigrid algorithm.

These linearized methods trade robustness and variational guarantee of con-

$$A^h \sim \frac{1}{3}\begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix} \qquad B^h \sim \frac{h^2}{36}\begin{pmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$

FIG. 1. Discretization stencils.

vergence for slightly simpler programming and smaller operation count. As we expected, however, the asymptotic convergence factors are quite similar. We demonstrate this in the next section.

6. NUMERICAL RESULTS

We coded a unigrid version of RQMG according to the description in Section 3 and applied it to the model problem in (1.3). Discretization used square elements and piecewise bilinear functions, resulting in the stencils depicted in Fig. 1. In each case reported here we discretized the domain Ω by a 33^2 uniform grid so that $h = \frac{1}{32}$. We let $m = 4$ so that the interior of Ω^1 consists of the single point $(\frac{1}{2}, \frac{1}{2})$. The f-support was centered at this midpoint and varied in grid size from 3^2 to 7^2 as indicated in Tables I and II. In each case we used red–black coordinate relaxation and a $V(0, 1)$-cycle. (Thus, the convergence factors for a full $V(1, 1)$-cycle should be approximately the square of that reported here (c.f., [8]), that is, about 0.06.) We report here on the asymptotic convergence factor, which we take to be the ratio between the successive residual norms $\|Au - RQ(u)Bu\|$ after many cycles. (Before computing these residual norms, u is normalized so that $\langle Bu, u \rangle = 1$.)

In Table I, all grids are global. These results indicate the insensitivity of the convergence factors to the size of the f-support. In Table II, the f-support is always 3^2 but the local grids vary in number from 0 to 1 to 2. All local grids are refinements of the finest global grid (33^2), by a factor of 2, and each is of the exact size of the f-support. Note that the convergence factors are essentially the same in all cases reported here. This is representative of our experience so far with RQMG when local grids are present.

To compare RQMG with the linearized version described in Section 5, we

TABLE I

RQMG: Insensitivity of Convergence Factors to Size of f-support

Grid size	f-support size	Number of local grids	Asymptotic Convergence factor
33^2	5^2	None	0.23
33^2	7^2	None	0.23
33^2	15^2	None	0.24

TABLE II

RQMG: Insensitivity of Convergence Factors to Number of Local Grids

Grid size	f-support size	Number of local grids	Asymptotic convergence factor
33^2	3^2	None	0.23
33^2	3^2	One	0.23
33^2	3^2	Two	0.24

applied both methods to (1.3) using the same basic components: $m=4$, the 33^2 uniform grid, the nine-point stencils in Fig. 1 and the associate intergrid transfers, red–black relaxation and a $V(0, 1)$-cycle, full f-support (i.e., 33^2), no local grids, and the initial guess obtained from $u(x, y)=\sin(1.5\pi x)\ \cos(1.5\pi y)$. Table III contains their convergence histories in terms of the eigenvalues and residual norms. Note the dramatic difference in performance of these two methods, especially away from the solution, The RQMG eigenvalue approximations converge in three iterations to six-digit accuracy, while the linearization takes eight. A similar rate comparison holds for the residuals. In fact, RQMG converges so fast that it experiences roundoff error effects by the seventh iteration. (These tests were performed in single precision Fortran on a Sequent Balance 2000; simple mechanisms can be used to circumvent the limiting effects of this precision on the iterative process, but none were used for the present set of experiments.) Finally, as expected, the eigenvalue convergence rates appear to be roughly the square of the residual convergence rates for both methods.

TABLE III

RQMG vs. Linearization

	Eigenvalues		Residuals	
Iteration	RQMG	Linearization	RQMG	Linearization
0	177.5611	177.5611	3.6 E2	3.6 E2
1	19.7638	92.1348	3.3 E0	1.0 E2
2	19.7552	34.1483	5.4 E−1	3.6 E1
3	19.7551	29.3944	1.0 E−1	1.8 E1
4	19.7551	21.1007	1.9 E−2	1.0 E1
5	19.7551	19.8029	3.4 E−3	1.9 E0
6	19.7551	19.7587	6.0 E−4	5.0 E−1
7	19.7551	19.7554	1.8 E−4	1.4 E−1
8	19.7751	19.7551	1.7 E−4	3.9 E−2

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